THE EULER-MARUYAMA APPROXIMATIONS FOR THE CEV MODEL.

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Abstract. The CEV model is given by the stochastic differential equation $X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma(X_s^+)^p dW_s$, $\frac{1}{2} \le p < 1$. It features a non-Lipschitz diffusion coefficient and gets absorbed at zero with a positive probability. We show the weak convergence of Euler-Maruyama approximations X_t^n to the process X_t, $0 \le t \le T$, in the Skorokhod metric. We give a new approximation by continuous processes which allows to relax some technical conditions in the proof of weak convergence in [19] done in terms of discrete time martingale problem. We calculate ruin probabilities as an example of such approximation. We establish that the ruin probability evaluated by simulations is not guaranteed to converge to the theoretical one, because the point zero is a discontinuity point of the limiting distribution. To establish such convergence we use the Levy metric, and also confirm the convergence numerically. Although the result is given for the specific model, our method works in a more general case of non-Lipschitz diffusion with absorbtion.

1. Introduction and the Main Result.

We consider the Constant Elasticity of Variance (CEV) model (e.g. [4], [5]) defined by the Itô equation with respect to a Brownian motion W_t

$$X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma(X_s^+)^p dW_s, \tag{1.1}$$

where $x^+ = 0 \vee x$, constants $p \in \left[\frac{1}{2}, 1\right)$, $\mu \in \mathbb{R}$, $\sigma > 0$, and a positive initial condition X_0. Denote by τ the first time of hitting zero $\tau = \inf\{t : X_t = 0\}$. It is known that zero is an absorbing state, e.g. [18], and that $\mathsf{P}(\tau < \infty) > 0$.

The aim of this paper is to prove validity of approximations of expectations of some functionals of these diffusions by the Euler-Maruyama scheme. For example, in finance this model represents a price process and it is important to evaluate expectations of payoffs depending on past prices $\mathsf{E}\mathcal{G}(X_{[0,T]})$; as well as for evaluation of the ruin probability $\mathsf{P}(\tau \le T)$ by simulations. In population modeling the diffusion with $p = \frac{1}{2}$ represents the size of a population and is known as Feller's branching diffusion. In this case τ represents the time to extinction of the population.

When exact theoretical expressions of such functionals exist then a comparison of simulated and exact results show how good the approximation is. However, in many cases exact expressions are not available. Then one needs to justify such approximations. Weak convergence established below assures convergence of expectations of bounded and continuous functionals of simulated values to the exact ones.

To allow a larger family of functionals to be approximated we show the weak convergence of approximations in the Skorokhod metric rather than uniform. We also

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consider approximation of the ruin probability $P(\tau \le t)$, which is the expectation of the past dependent and discontinuous in this metric functional $I_{\{\tau\le T\}}$.

The fact that the diffusion coefficient is singular and non-Lipschitz makes the analysis non-standard. Feller in [6] showed for the case $p = 1/2$ that the solution of the Fokker-Plank equation exists and is unique and gave its fundamental solution. This fact is used for evaluation of ruin probability by claiming that for any T, X_T has a density on $(0, \infty)$, and in particular its distribution function is continuous at any positive point.

Previous various results on the Euler-Maruyama algorithm for different type of diffusion models can be found in Kloeden and Platten [14] and Milstein and Tretyakov [16]. They are concerned mostly with the approximation of the final value X_T, rather than the whole trajectory X_t, $0 \le t \le T$. When both the drift and diffusion coefficients are Lipschitz and the diffusion coefficient is nonsingular, the standard theory applies. There is a body of literature on the topic of approximations, see e.g. Bally and Talay [1], Bossy and Diop [3], Gyöngy [7], Gyöngy and Krylov [8], Halidias and Kloeden [9], Higham, Mao and Stuart [11], Hutzenthaler and Jentzen [12]. Zähle [19] considers the case when the classical "Lipschitz-Lipschitz" setting fails. However, as a rule, the cases considered in the literature are such that the drift and diffusion coefficients exclude the absorbtion effect even when the diffusion coefficient is singular such as in Bessel diffusion.

We chose to give the results for the particular model of CEV for the sake of transparency, and because this model is of significance in applications. The reader will note that the proofs are sufficiently involved already in this special case. However, our method and result hold in a more general case of non-Lipschitz diffusion with absorbtion.

Recall next the Euler-Maruyama scheme for the diffusion process X_t. Taking for simplicity equidistant partitions of $[0, T]$, $0 \equiv t^n_0 < t^n_1 < \ldots < t^n_{t_{n-1}} < t^n_n \equiv T$, $t^n_k - t^n_{k-1} \equiv \frac{T}{n}$, it is defined by the following recursion

$$\begin{aligned}
X^n_{t^n_0} &= X_0, \\
X^n_{t^n_k} &= X^n_{t^n_{k-1}} + \mu \frac{T}{n}(X^n_{t^n_{k-1}})^+ + \sigma(X^{n+}_{t^n_{k-1}})^p \sqrt{\frac{T}{n}}\xi_k, \\
X^n_t &= X^n_{t^n_{k-1}}, \quad t \in [t^n_{k-1}, t^n_k), \quad k = 1, \ldots n,
\end{aligned} \tag{1.2}$$

where $(\xi_k)_{k\ge1}$ is i.i.d. sequence of $(0, 1)$-Gaussian random variables.

The process $X^n_{t^n_k}$ can be considered as a discrete time semimartingale (see, e.g., Zähle [19]) and X^n_t as a continuous time semimartingale with discontinuous paths. It may seem that there is not much difference between discrete or continuous time approximations. However, using a continuous time continuous approximation of X^n_t introduced in the sequel enables us to simplify the proof of the weak convergence and avoid some technical assumptions used in [19], e.g. (2.3) ibid.

The paths of the process $X^n = (X^n_t)_{t\in[0,T]}$ are right continuous piece-wise constant functions with left limits belonging to the Skorokhod space $\mathbb{D} = \mathbb{D}_{[0,T]}$. Consider a metric space $(\mathbb{D}, d_0)$ endowed with the Skorokhod metric d_0: if $\boldsymbol{x}$, $\boldsymbol{y} \in \mathbb{D}$, then

$$d_0(\boldsymbol{x}, \boldsymbol{y}) = \inf_{\varphi\in\Phi} \Big\{ \sup_{t\in[0,T]} |\boldsymbol{x}_t - \boldsymbol{y}_{\varphi(t)}| + \sup_{0\le s<t\le 1} \Big| \log \frac{\varphi(t) - \varphi(s)}{t - s} \Big| \Big\},$$

where Φ is a set of strictly increasing continuous functions $\varphi = (\varphi(t))_{0\le t\le T}$ with $\varphi(0) = 0$, $\varphi(T) = 1$.

Denote by Q^n the probability measure on the space $(\mathbb{D}, \mathscr{D})$, where $\mathscr{D}$ is the Borel σ-algebra, of the distribution of $X^n = (X^n_t)_{t\in[0,T]}$, and by Q the distribution of

$X = (X_t)_{t\in[0,T]}$. Since X is a continuous process, $\mathsf{Q}(\mathbb{C}) = 1$, where $\mathbb{C}$ denotes the subspace of continuous functions.

Evaluations of functionals by simulations is justified by the convergence of measures in the Skorokhod space $\mathsf{Q}^n \xrightarrow[n\to\infty]{\mathrm{d}_0} \mathsf{Q}$, that is,

$$\lim_{n\to\infty} \int_{\mathbb{D}} f(\boldsymbol{x}) d\mathsf{Q}^n = \int_{\mathbb{C}} f(\boldsymbol{x}) d\mathsf{Q}$$

for any bounded and continuous in the metric d_0 function $f(\boldsymbol{x})$. The aim of the paper is precisely to show this convergence. Note that such a function $f(\boldsymbol{x})$ need not be continuous in the uniform metric $\varrho(\boldsymbol{x}, \boldsymbol{y}) = \sup\limits_{t\in[0,T]} |\boldsymbol{x}_t - \boldsymbol{y}_t|$.

We remark on the use of the function x^+ next. Since the process $X = (X_t)_{t\in[0,T]}$ is nonnegative, the notation X_t^+ in (1.1) is used formally. However, it must be used for the approximating process X_t^{n+} in (1.2), because X_t^n can (and does) become negative. Since X_t^n has piece-wise constant paths the stopping time

$$\tau^n = \inf\{t \le T : X_t^n \le 0\}$$

much more likely corresponds to the negative value rather than the zero value of $X_{\tau_n}^n$. The plus notation X_t^{n+} in (1.2) enables us to exclude negative values of X_t^n everywhere except of $X_{\tau_n}^n$. Thus, in simulations of $f(X)$ we must use $f(X^{n+})$. Next, the function $g(\boldsymbol{x}) := f(\boldsymbol{x}^+)$ inherits the properties of $f(\boldsymbol{x})$ and is bounded and continuous in the metric d_0, moreover, when applied to simulations it converges to the desired limit

$$\lim_{n\to\infty} \mathsf{E} f(X^{n+}) \equiv \lim_{n\to\infty} \mathsf{E} g(X^n) = \mathsf{E} g(X) = \mathsf{E} f(X^+) \equiv \mathsf{E} f(X).$$

We introduce now the continuous approximation $\widetilde{X}_t^n$ used in the proof of weak convergence

$$\widetilde{X}_t^n = X_0 + \sum_{k=1}^n \int_{t_{k-1}^n}^{t\wedge\tau_k^n} \mu(\widetilde{X}_{t_{k-1}^n}^n)^+ ds + \sum_{k=1}^n \int_{t_{k-1}^n}^{t\wedge t_k^n} \sigma(\widetilde{X}_{t_{k-1}^n}^{n+})^p d\widetilde{W}_s,$$

where $\widetilde{W}_t$ is a Brownian motion such that $\widetilde{W}_{t_k^n} - \widetilde{W}_{t_{k-1}^n} \equiv \sqrt{\frac{T}{n}}\xi_k$.

Denote $\widetilde{\mathsf{Q}}^n$ the distribution of $(\widetilde{X}_t^n)_{t\in[0,T]}$. The main result is that $\mathsf{Q}^n \xrightarrow[n\to\infty]{\mathrm{d}_0} \mathsf{Q}$.

Theorem 1.1. *For any $T > 0$, the Euler-Maruyama approximation for the model* (1.2) *converges weakly in the Skorokhod metric d_0 to the limit process $(X_t)_{t\in[0,T]}$ defined in* (1.1).

The proof is done in three steps: first we show that the distance in the uniform metric between the Euler-Maruyama and the continuous semimartingale approximations converges to zero in probability, secondly we show that the continuous approximation converges in the Skorokhod metric to the solution, and finally we deduce convergence of the Euler-Maruyama approximation. Thus the three steps in the proof are

$$\begin{aligned}
&\textbf{step 1.} \quad \varrho(X^n, \widetilde{X}^n) \xrightarrow[n\to\infty]{\text{prob.}} 0 \\
&\textbf{step 2.} \quad \widetilde{\mathsf{Q}}^n \xrightarrow[n\to\infty]{\mathrm{d}_0} \mathsf{Q} \\
&\textbf{step 3.} \quad \left.\begin{matrix} \varrho(X^n, \widetilde{X}^n) \xrightarrow[n\to\infty]{\text{prob.}} 0 \\ \widetilde{\mathsf{Q}}^n \xrightarrow[n\to\infty]{\mathrm{d}_0} \mathsf{Q} \end{matrix}\right\} \Rightarrow \mathsf{Q}^n \xrightarrow[n\to\infty]{\mathrm{d}_0} \mathsf{Q}.
\end{aligned} \tag{1.3}$$

The proof of "step 1" uses standard stochastic calculus calculations for semimartingales. The proof of "step 2" follows from the results on diffusion approximation for semimartingales (see [15, Ch.8]). The proof of "step 3" is done as a standard application of Billingsley [2, Theorem 4.4, Ch.1, §4].

The paper is organized as follows. In Section 2 we give preliminary results used in the proof, with some being of independent interest. The proof of the Theorem is given in Section 3. Section 4 gives simulations for the ruin probability $\mathsf{P}(\tau < T)$. To be self contained, the existence and uniqueness of solutions of the stochastic differential equation for the CEV model (1.1) is ahown in Section 5.

2. **Preliminaries**

We shall use the Doob maximal inequality for martingales (see e.g. [15, Ch.1, §9] and [13, p.201]) in the following context. Let $(\alpha_k)_{0\le k\le n-1}$ and $(\xi_k)_{1\le k\le n}$ be random variables such that
- $\mathsf{E}\alpha_k^2 < \infty$, $0 \le k \le n-1$;
- $(\xi_k)_{1\le k\le n}$ is i.i.d. with $\mathsf{E}\xi_1 = 0$, $\mathsf{E}\xi_1^2 = 1$;
- ξ_k and $\{\alpha_0, \dots, \alpha_{k-1}\}$ are independent for any $1 \le k \le n$.

Set $M_t = \sum_{j=1}^{[nt]} \alpha_{j-1}\xi_j$, where $t \in [0,1]$ and $[nt] = k$ if $t \in (\frac{k-1}{n}, \frac{k}{n}]$. The process M_t is a square integrable martingale for a suitable filtration. So, the Doob maximal inequality $\mathsf{E}\big(\big|\sup_{t\in[0,1]} M_t\big|^2\big) \le 4\mathsf{E}M_1^2$ is equivalent to

$$\mathsf{E}\max_{1\le k\le n}\Big|\sum_{j=1}^{k}\alpha_{j-1}\xi_j\Big|^2 \le 4\sum_{j=1}^{n}\mathsf{E}\alpha_{j-1}^2. \tag{2.1}$$

Next we give the result useful in the analysis of products of random variables.

Lemma 2.1. *Let X_n, Y_n be sequences of random variables such that X_n converges to zero in probability and expectations of Y_n are uniformly bounded,* $\sup_n \mathsf{E}|Y_n| = \mathbf{r} < \infty$. *Then X_nY_n converges to zero in probability.*

Proof.

$$\begin{aligned}\mathsf{P}\big(|X_nY_n| > \varepsilon\big) &= \mathsf{P}\big(|Y_n| \le C, |X_nY_n| > \varepsilon\big) + \mathsf{P}\big(|Y_n| > C, |X_nY_n| > \varepsilon\big)\\ &\le \mathsf{P}\big(C|X_n| > \varepsilon\big) + \mathsf{P}\big(|Y_n| > C\big).\end{aligned}$$

By the Chebyshev inequality $\mathsf{P}\big(|Y_n| > C\big) \le \frac{\mathbf{r}}{C}$. Hence

$$\varlimsup_{n\to\infty}\mathsf{P}\big(|X_nY_n| > \varepsilon\big) \le \frac{\mathbf{r}}{C} \xrightarrow[C\to\infty]{} 0.$$

□

Next we need the following result of convergence to zero the expectation of double-maximum the Brownian motion increments over partitions.

Lemma 2.2. *Let W_t be a Brownian motion and $\{t_i\}$, $i = 1, \dots, n$ an equidistant partition of $[0,1]$. Let $W_i^* = \sup_{t\in[t_{i-1},t_i]}|W_t - W_{t_{i-1}}|$, and $M_n = \max_{1\le i\le n} W_i^*$. Then M_n converges to zero in $\mathbb{L}^k$ for any $k \ge 1$.*

Proof. We calculate k-th moment of M_n, $k \ge 1$. Since $M_n \ge 0$

$$\begin{aligned}\mathsf{E}M_n^k = \int_0^\infty \mathsf{P}\big(M_n^k \ge x\big)dx &= \int_0^\infty \mathsf{P}\big(M_n \ge x^{1/k}\big)dx\\ &\le n\int_0^\infty \mathsf{P}\Big(W_1^* \ge x^{1/k}\Big)dx.\end{aligned}$$

The inequality is obtained since $(W_i^*)_{1\le i\le n}$ are i.i.d. random variables, therefore for any $x \ge 0$

$$\mathsf{P}\big(M_n \ge x^{1/k}\big) = \mathsf{P}\Big(\max_{1\le i\le n} W_i^* \ge x^{1/k}\Big) = \mathsf{P}\Big(\bigcup_{i=1}^{n}\{W_i^* \ge x^{1/k}\}\Big) \le n\mathsf{P}\Big(W_1^* \ge x^{1/k}\Big).$$

Next since $\sup_{t\in[0,1/n]}|W_t| \le \sup_{t\in[0,1/n]} W_t + \sup_{t\in[0,1/n]}(-W_t)$, it follows

$$\mathsf{P}\Big(W_1^* \ge x^{1/k}\Big) \le \mathsf{P}\Big(\sup_{t\in[0,1/n]} W_t + \sup_{t\in[0,1/n]}(-W_t) \ge x^{1/k}\Big)$$
$$\le \mathsf{P}\Big(\sup_{t\in[0,1/n]} W_t \ge \frac{1}{2}x^{1/k}\Big) + \mathsf{P}\Big(\sup_{t\in[0,1/n]}(-W_t) \ge \frac{1}{2}x^{1/k}\Big) = 2\mathsf{P}\Big(W_{1/n} \ge \frac{1}{2}x^{1/k}\Big),$$

where we have used the well-known law of the maximum of Brownian motion.

Thus

$$\mathsf{E}M_n^k \le 4n\int_0^\infty \mathsf{P}\Big(W_{1/n} \ge \frac{1}{2}x^{1/k}\Big)dx \le 4n\int_0^\infty \mathsf{P}\Big(|W_{1/n}| \ge \frac{1}{2}x^{1/k}\Big)dx$$
$$= 4n2^k\mathsf{E}|W_{1/n}|^k = C_k n^{1-k/2},$$

where C_k depends only on k, ($C_k = 2^{k+2}\mathsf{E}|\xi|^k$ with $\xi \sim N(0,1)$). Hence M_n converges to zero in $\mathbb{L}^k$ for any $k > 2$. But since convergence in $\mathbb{L}^p$ for a $p > 1$ implies convergence in $\mathbb{L}^k$ for any $k \in [1,p]$, the statement is proved. □

The next elementary inequality is new and is instrumental in the proof.

Lemma 2.3. *For any $x, y \in \mathbb{R}$, and $p \in [\frac{1}{2}, 1)$,*

$$|(x^+)^{2p} - (y^+)^{2p}| \le \begin{cases} |x-y|, & p = \frac{1}{2} \\ (2+|x|+|y|)|x-y|^p, & p \in \big(\frac{1}{2}, 1\big). \end{cases}$$

Proof. For $p = \frac{1}{2}$ we have

$$\begin{aligned} |x^+ - y^+| &= |x-y|I_{\{x>0,y>0\}} + |x|I_{\{x>0,y\le 0\}} + |y|I_{\{x\le 0,y>0\}} \\ &\le |x-y|I_{\{x>0,y>0\}} + |x-y|I_{\{x>0,y\le 0\}} + |y-x|I_{\{x\le 0,y>0\}} \le |x-y|. \end{aligned} \tag{2.2}$$

For $p \in \big(\frac{1}{2}, 1\big)$ we have

$$\begin{aligned} |(x^+)^{2p} - (y^+)^{2p}| &= |(x^+)^p - (y^+)^p||(x^+)^p + (y^+)^p| \\ &\le (2+|x|+|y|)|(x^+)^p - (y^+)^p|. \end{aligned}$$

Next we now show that

$$|x^p - y^p| \le |x-y|^p.$$

For $x = y = 0$ it is obvious.

Consider $x > 0$ and $y < 0$. Then, taking into account the proved inequality $|x^+ - y^+| \le |x-y|$ (the statement of this lemma for $p = \frac{1}{2}$) and the fact that $y^+ = 0$, we obtain

$$|(x^+)^p - (y^+)^p| = |x^+|^p = |x^+ - y^+|^p \le |x-y|^p.$$

Clearly, the inequality remains true for $x < 0$ and $y > 0$.

If both x and y are positive and $x > y$, then

$$|x^p - y^p|^{1/p} = x\Big|1 - \Big(\frac{y}{x}\Big)^p\Big|^{1/p} \le x\Big|1 - \Big(\frac{y}{x}\Big)\Big|^{1/p} \le x\Big|1 - \Big(\frac{y}{x}\Big)\Big| = |x-y|$$

and, in turn, $|x^p - y^p| \le |x-y|^p$. It is easy to see the inequality remains true for $y > x$. □

3. **Proof of Theorem 1.1**

In the following result we show that the maximum of discrete approximations have uniformly bounded seconds moments. Henceforth $\mathbf{r}$ denotes a generic positive constant independent of n with different values at different appearances.

Lemma 3.1. *Let $X^n_{t^n_k}$ be the Euler-Maruyama approximation defined in* (1.2). *Then* $\mathsf{E}\max\limits_{1\le k\le n}|X^n_{t^n_k}|^2\le \mathbf{r}$.

Proof. First we bound from above $\max\limits_{1\le k\le n}\mathsf{E}|X^n_{t^n_k}|^2$. By (1.2)

$$\mathsf{E}|X^n_{t^n_k}|^2=\mathsf{E}|X^n_{t^n_{k-1}}|^2\Big(1+\frac{\mu T}{n}\Big)^2+\sigma^2\mathsf{E}(X^{n+}_{t^n_{k-1}})^{2p}\frac{T}{n}. \tag{3.1}$$

We use repeatedly the following bound of $(X^{n+}_{t^n_{k-1}})^{2p}$. Since $(x^+)^{2p}\le |x|^{2p}$ and $2p<2$, we have $(x^+)^{2p}\le 1+x^2$. Hence

$$\mathsf{E}(X^{n+}_{t^n_{k-1}})^{2p}\le 1+\mathsf{E}|X^n_{t^n_{k-1}}|^2. \tag{3.2}$$

Next, for sufficiently large n, there exists $\mathbf{r}$ such that

$$\big(1+\frac{\mu T}{n}\big)^2\le 1+\frac{\mathbf{r}}{n}. \tag{3.3}$$

Hence for sufficiently large n, we obtain from (3.1) by using (3.3) the recurrent inequality

$$\begin{aligned}\mathsf{E}|X^n_{t^n_k}|^2&\le \mathsf{E}|X^n_{t^n_{k-1}}|^2\Big(1+\frac{\mathbf{r}}{n}\Big)+\frac{\sigma^2T}{n}\Big(1+\mathsf{E}|X^n_{t^n_k}|^2\Big)\\ &=\mathsf{E}|X^n_{t^n_{k-1}}|^2\Big(1+\frac{\mathbf{r}+\sigma^2T}{n}\Big)+\frac{\sigma^2T}{n}:=\mathsf{E}|X^n_{t^n_{k-1}}|^2\Big(1+\frac{\mathbf{r}}{n}\Big)+\frac{\mathbf{r}}{n}.\end{aligned}$$

Iterating it, for $k\le n$ we obtain

$$\begin{aligned}\mathsf{E}|X^n_{t^n_k}|^2&\le X_0^2\Big(1+\frac{\mathbf{r}}{n}\Big)^k+\frac{r}{n}\sum_{j=1}^k\Big(1+\frac{\mathbf{r}}{n}\Big)^{k-j+1}\\ &\le (X_0^2+\mathbf{r})\Big(1+\frac{\mathbf{r}}{n}\Big)^n=(X_0^2+\mathbf{r})O\big(e^{\mathbf{r}}\big).\end{aligned}$$

Thus

$$\max_{1\le k\le n}\mathsf{E}|X^n_{t^n_k}|^2\le \mathbf{r}. \tag{3.4}$$

From the definition of the scheme (1.2) we obtain by iterations

$$\max_{1\le k\le n}|X^n_{t^n_k}|\le |X_0|+|\mu|\frac{T}{n}\sum_{j=1}^n|X^n_{t^n_{j-1}}|+\sqrt{\frac{T}{n}}\max_{1\le k\le n}\Big|\sum_{j=1}^k\sigma(X^{n+}_{t^n_{j-1}})^p\xi_j\Big|.$$

Next we use the Cauchy-Schwarz inequality $(\sum_{i=1}^l a_i)^2\le l\sum_{i=1}^l a_i^2$ with $l=3$. Proceeding from above we have

$$\begin{aligned}&\mathsf{E}\max_{1\le k\le n}|X^n_{t^n_k}|^2\\ &\quad\le 3|X_0|^2+3\frac{\mu^2T^2}{n^2}\mathsf{E}\Big(\sum_{j=1}^n|X^n_{t^n_{j-1}}|\Big)^2+3\frac{T}{n}\mathsf{E}\max_{1\le k\le n}\Big|\sum_{j=1}^k\sigma(X^{n+}_{t^n_{j-1}})^p\xi_j\Big|^2. \end{aligned}\tag{3.5}$$

The Cauchy-Schwarz inequality $\Big(\sum_{j=1}^n|X^n_{t^n_{j-1}}|\Big)^2\le n\sum_{j=1}^n(X^n_{t^n_{j-1}})^2$ coupled with the bound on the largest second moment shown above (3.4) implies that the second

term in (3.5) is bounded by a constant $\mathbf{r}$ independent of n. The last term is bounded by using the Doob's maximal inequality (2.1) and the bound in (3.2)

$$\mathsf{E}\max_{1\le k\le n}\Big|\sum_{j=1}^{k}\sigma(X^{n+}_{t^n_{j-1}})^p\xi_j\Big|^2 \le 4\sigma^2\sum_{j=1}^{n}\mathsf{E}(X^n_{t^n_{j-1}})^{2p} \le 4\sigma^2\sum_{j=1}^{n}[1+\mathsf{E}(X^n_{t^n_{j-1}})^2].$$

Using (3.4) we can now see that the last term in (3.5) is bounded by $\mathbf{r}$. □

We proceed now to prove the steps in (1.3).

3.1. **Step 1.** The distance between the two approximations in the sup norm converges to zero in probability and in $\mathbb{L}^2$.

Lemma 3.2. $\mathsf{E}\varrho^2(X^n,\widetilde{X}^n) \xrightarrow[n\to\infty]{} 0.$

Proof. From the definitions of X^n_t and $\widetilde{X}^n_t$ it follows that at the points of the partitions both approximations coincide and at intermediate points the following holds

$$\begin{aligned} X^n_{t^n_k} &\equiv \widetilde{X}^n_{t^n_k}\\ \widetilde{X}^n_t - X^n_{t^n_{k-1}} &= \int_{t^n_{k-1}}^{t\wedge t^n_k}\mu(X^n_{t^n_{k-1}})^+ds + \int_{t^n_{k-1}}^{t\wedge t^n_k}\sigma(X^{n+}_{t^n_{k-1}})^p d\widetilde{W}_s, \quad t\in[t^n_{k-1},t^n_k]. \end{aligned} \tag{3.6}$$

By the formula (3.6),

$$\sup_{t\in[t^n_{k-1},t^n_k]}|\widetilde{X}^n_t - X^n_{t^n_{k-1}}| \le \frac{|\mu|T}{n}|X^n_{t^n_{k-1}}| + \sigma\sup_{t\in[t^n_{k-1},t^n_k]}\big|\widetilde{W}_t-\widetilde{W}_{t^n_{k-1}}\big||X^n_{t^n_{k-1}}|^p.$$

Consequently,

$$\begin{aligned}\varrho(X^n,\widetilde{X}^n) &= \sup_{t\in[0,T]}|\widetilde{X}^n_t - X^n_t| = \max_{1\le k\le n}\sup_{t\in[t^n_{k-1},t^n_k]}|\widetilde{X}^n_t - X^n_{t^n_{k-1}}|\\ &\le \frac{|\mu|T}{n}\max_{1\le k\le n}|X^n_{t^n_{k-1}}| + \sigma\max_{1\le k\le n}\sup_{t\in[t^n_{k-1},t^n_k]}\big|\widetilde{W}_t-\widetilde{W}_{t^n_{k-1}}\big|\max_{1\le k\le n}|X^n_{t^n_{k-1}}|^p.\end{aligned}$$

The first term converges to zero in $\mathbb{L}^2$ using Lemma 3.1 as $\mathsf{E}\max_{1\le k\le n}|X^n_{t^n_{k-1}}|\le\mathbf{r}$. For the second term use Hölder's inequality with parameters $\frac{2}{p}$ and $\frac{2}{2-p}$:

$$\begin{aligned}&\mathsf{E}\Big(\max_{1\le k\le n}\sup_{t\in[t^n_{k-1},t^n_k]}\big|\widetilde{W}_t-\widetilde{W}_{t^n_{k-1}}\big|\max_{1\le k\le n}|X^n_{t^n_{k-1}}|^p\Big)^2\\ &\le\bigg(\mathsf{E}\Big[\max_{1\le k\le n}\sup_{t\in[t^n_{k-1},t^n_k]}\big|\widetilde{W}_t-\widetilde{W}_{t^n_{k-1}}\big|\Big]^{\frac{2}{2-p}}\bigg)^{2-p}\Big(\mathsf{E}\max_{1\le k\le n}|X^n_{t^n_{k-1}}|^2\Big)^p.\end{aligned}$$

By Lemma 3.1 $\sup_n\mathsf{E}\max_{1\le k\le n}|X^n_{t^n_{k-1}}|^2\le\mathbf{r}$. Since $\frac{2}{2-p}>1$, by Lemma 2.2

$$\lim_{n\to\infty}\mathsf{E}\Big(\max_{1\le k\le n}\sup_{t\in[t^n_{k-1},t^n_k]}\big|\widetilde{W}_t-\widetilde{W}_{t^n_{k-1}}\big|\Big)^{2/(2-p)}=0.$$

□

3.2. **Step 2.** Weak convergence of continuous approximations.

Lemma 3.3. $\widetilde{\mathsf{Q}}^n \xrightarrow[n\to\infty]{\mathrm{d}_0} \mathsf{Q}.$

Proof. The proof rests on a general result on the weak convergence of semimartingales to a diffusion [15], Theorem 1, Ch. 8, §3. This theorem states that for weak convergence to a diffusion it is enough to check convergence of the drifts and quadratic variations evaluated at the pre-limit processes. The processes X and $\widetilde{X}^n$ are semimartingales with following decompositions

$$\begin{aligned} X_t &= X_0 + \underbrace{\int_0^t \mu X_s ds}_{:=B_t(X)} + \underbrace{\int_0^t \sigma(X_s^+)^p dW_s}_{:=M_t(X)} \\ \widetilde{X}_t^n &= X_0 + \underbrace{\sum_{k=1}^n \int_{t_{k-1}^n}^{t\wedge t_k^n} \mu \widetilde{X}^{n+}_{t_{k-1}^n} ds}_{:=B_t^n(\widetilde{X}^n)} + \underbrace{\sum_{k=1}^n \int_{t_{k-1}^n}^{t\wedge t_k^n} \sigma(\widetilde{X}^{n+}_{t_{k-1}^n})^p d\widetilde{W}_s}_{:=M_t^n(\widetilde{X}^n)}, \end{aligned}$$

where we have denoted above

(B) $B_t(X)$ and $B_t^n(\widetilde{X}^n)$ are drifts

(M) $M_t(X)$ and $M_t^n(\widetilde{X}^n)$ are continuous martingales with predictable quadratic variations

$$\begin{aligned} \langle M\rangle_t(X) &= \int_0^t \sigma^2(X_s^+)^{2p} ds, \\ \langle M^n\rangle_t(\widetilde{X}^n) &= \sum_{k=1}^n \int_{t_{k-1}^n}^{t\wedge t_k^n} \sigma^2(\widetilde{X}^{n+}_{t_{k-1}^n})^{2p} ds. \end{aligned}$$

The above mentioned Theorem 1 of [15], adapted to the present setting, states that the weak convergence takes place if the following three conditions hold.

$$\left.\begin{array}{ll} (\mathbf{a}) & \text{Equation (1.1) has a unique (at least weak) solution} \\ (\mathbf{b}) & \sup\limits_{t\in[0,T]} |B_t(\widetilde{X}^n) - B_t^n(\widetilde{X}^n)| \xrightarrow[n\to\infty]{\text{prob.}} 0 \\ (\mathbf{c}) & \sup\limits_{t\in[0,T]} |\langle M\rangle(\widetilde{X}^n) - \langle M^n\rangle_t(\widetilde{X}^n)| \xrightarrow[n\to\infty]{\text{prob.}} 0, \end{array}\right\} \Rightarrow \widetilde{\mathsf{Q}}^n \xrightarrow[n\to\infty]{\mathrm{d}_0} \mathsf{Q}.$$

We proceed to verify these conditions. The existence and uniqueness of (1.1) is known, e.g. [18], and is also given for completeness Proposition 5.1. Hence **(a)** holds. To show **(b)** write, taking into account (2.2),

$$\sup_{t\in[0,T]} \Big| B_t(\widetilde{X}^n) - B_t^n(\widetilde{X}^n)] \Big| \le |\mu| \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} |\widetilde{X}_s^n - \widetilde{X}^n_{t_{k-1}^n}| ds \le T|\mu| \varrho(\widetilde{X}^n, X^n).$$

Hence **(b)** holds by applying Lemma 3.2.

To prove **(c)** write the bound

$$\begin{aligned} \sup_{t\in[0,T]} \Big| \langle M\rangle_t(\widetilde{X}^n) - \langle M^n\rangle_t(\widetilde{X}^n) \Big| &\le \sigma^2 \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} |(\widetilde{X}_s^{n+})^{2p} - (\widetilde{X}^{n+}_{t_{k-1}^n})^{2p}| ds \\ &= \sigma^2 \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} |(\widetilde{X}_s^{n+})^{2p} - (X^{n+}_{t_{k-1}^n})^{2p}| ds, \end{aligned} \tag{3.7}$$

where we have used that the two approximations coincide on the grid. By applying Lemma 2.3 we have further bound on the expression under the integral

$$
\begin{aligned}
&|(\widetilde{X}_s^{n+})^{2p} - (X_{t_{k-1}^n}^{n+})^{2p}| \\
&\quad \le \begin{cases} |\widetilde{X}_s^n - X_{t_{k-1}^n}^n|, & p = \frac{1}{2} \\ \Big(2 + \sup\limits_{t\in[0,T]} |\widetilde{X}_t^n| + \sup\limits_{t\in[0,T]} |X_t^n|\Big) \sup\limits_{s\in[t_{k-1}^n, t_k^n]} |\widetilde{X}_s^n - X_{t_{k-1}^n}^n|^p, & p \in \big(\frac{1}{2}, 1\big) \end{cases}.
\end{aligned}
$$

Hence for $p = \frac{1}{2}$ the bound in (3.7) becomes

$$
\sigma^2 \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} |(\widetilde{X}_s^{n+})^{2p} - (X_{t_{k-1}^n}^{n+})^{2p}| ds \le \sigma^2 T \varrho(\widetilde{X}^n, X^n)
$$

and the statement follows by Lemma 3.2.

For $p \in (\frac{1}{2}, 1)$ the bound in (3.7) becomes

$$
\begin{aligned}
&\sigma^2 \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} |(\widetilde{X}_s^{n+})^{2p} - (X_{t_{k-1}^n}^{n+})^{2p}| ds = \sigma^2 \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} |(\widetilde{X}_s^{n+})^{2p} - (X_{t_{k-1}^n}^{n+})^{2p}| ds \\
&\le \sigma^2 \Big(2 + \sup_{t\in[0,T]} |\widetilde{X}_t^n| + \sup_{t\in[0,T]} |X_t^n|\Big) \sum_{k=1}^n \int_{t_{k-1}^n}^{t_k^n} \sup_{s\in[t_{k-1}^n, t_k^n]} |\widetilde{X}_s^n - X_s^n|^p ds \\
&\le T\sigma^2 \Big(2 + \sup_{t\in[0,T]} |\widetilde{X}_t^n| + \sup_{t\in[0,T]} |X_t^n|\Big) \varrho^p(\widetilde{X}^n, X^n).
\end{aligned}
$$

By Lemma 3.1, Lemma 3.2 we have the product of two terms, one of which has uniformly bounded expectations and the second converges in probability to zero. By Lemma 2.1 the product converges in probability to zero.

Thus the conditions of the Theorem 1 of [15] are verified and weak convergence is proved. □

3.3. **Step 3.** Weak convergence of the Euler-Maruyama approximations.

Lemma 3.4. *For any bounded and continuous in the metric d_0 function $f(\boldsymbol{x})$,*

$$
\overline{\lim_{n\to\infty}} |\mathsf{E} f(X^n) - \mathsf{E} f(X)| \le \overline{\lim_{n\to\infty}} |\mathsf{E} f(\widetilde{X}^n) - \mathsf{E} f(X)| = 0.
$$

Proof. The result follows from the triangular inequality

$$
|\mathsf{E} f(X^n) - \mathsf{E} f(X)| \le \mathsf{E}|f(X^n) - f(\widetilde{X}^n)| + |\mathsf{E} f(\widetilde{X}^n) - \mathsf{E} f(X)|. \tag{3.8}
$$

Since convergence in the uniform metric implies convergence in the Skorokhod metric, by Lemma 3.2 we have, since f is continuous in this metric,

$$
\lim_{n\to\infty} \mathsf{E}|f(X^n) - f(\widetilde{X}^n)| = 0.
$$

Taking now the lim sup in (3.8) and using the previous result of weak convergence of $\widetilde{X}^n$ to X proves the step 3. □

4. Evaluation of ruin probability on a finite time interval by simulations.

In this section we evaluate numerically a ruin probability $\mathsf{P}(\tau \le T)$ on a time interval $[0,T]$, where $\tau = \{t : X_t = 0\}$ by Euler-Maruyama approximations. An explicit formula to the ruin probability, as a function of arguments p, X_0, μ, σ, T, is known in the case $p = \frac{1}{2}, T = \infty$:

$$
\mathsf{P}(\tau < \infty) = \exp\Big(-\frac{2\mu}{\sigma^2} X_0 \Big)
$$

(see, e.g. [13, p. 354]), but not for finite T.

Naturally, $\mathsf{P}(\tau \le T)$ is important in applications, and we study it for different values of parameters p, X_0, μ, σ, T:

$$p = \begin{cases} \frac{1}{2} \\ \frac{3}{4}, \end{cases} \quad X_0 = \begin{cases} \frac{1}{10} \\ \frac{1}{4} \\ 1, \end{cases} \quad \mu = \begin{cases} 1 \\ -1, \end{cases} \quad \sigma = 1, T = \begin{cases} 3 \\ 9. \end{cases}$$

We combine the Euler-Maruyama simulation algorithm and the Monte-Carlo technique with 10^3 runs per point.

The basis for analysis is an obvious formula $\mathsf{P}(\tau \le T) = \mathsf{P}(X_T = 0)$. It allows us to deal with the distribution function $F(x) := \mathsf{P}(X_T \le x)$ of X_T instead of harder to compute distribution function of τ, $\mathsf{P}(\tau \le T)$. Notice that

$$F(x) = \begin{cases} 0, & x < 0 \\ F(0) = \mathsf{P}(X_T = 0) > 0, & x = 0, \end{cases}$$

that is, $F(0) - F(0-) > 0$ and so the distribution function $F(x)$ has an atom at the point $\{0\}$.

The measure Q is supported on the space of continuous functions. So the weak convergence of processes and measures $\mathsf{Q}^n \xrightarrow[n\to\infty]{\mathrm{d}_0} \mathsf{Q}$ implies weak convergence of finite dimensional distributions, and in particular marginals, $X_T^n \xrightarrow[n\to\infty]{\text{law}} X_T$. That is if $F_n(x) = \mathsf{P}(X_T^n \le x)$ then $\lim_{n\to\infty} F_n(x) = F(x)$ at any point of continuity of F. Unfortunately 0, our point of interest, is an atom of F and we can not claim that

$$\lim_{n\to\infty} F_n(0) = F(0) = \mathsf{P}(\tau \le T).$$

In view of this uncertainty, we give approximations for lower and upper bounds of $F(0)$ by using the Lévy metric (see e.g. [10]):

$$\mathcal{L}(F_n, F) = \inf\{h > 0 : F_n(x-h) - h \le F(x) \le F_n(x+h) + h;\ \forall\, x\}.$$

It is known that weak convergence of distributions implies convergence in the Lévy metric $\mathsf{Q}^n \xrightarrow[n\to\infty]{\mathrm{d}_0} \mathsf{Q} \Rightarrow \lim_{n\to\infty} \mathcal{L}(F_n, F) = 0$.

Though $\lim_{n\to\infty} \mathcal{L}(F_n, F) = 0$ does not fish out atom it helps localize it. Namely we take
$x = 0$ and a small suitable ε_n , in each case determined experimentally, such that the interval $[F_n(-\varepsilon_n) - \varepsilon_n, F_n(\varepsilon_n) + \varepsilon_n]$ is small enough and declare that an estimate of $F(0)$ belongs to this interval. Such values of ε_n are pointed out below as last values in each Table.

Simulations for various values of parameters show good fit of this procedure.

Example 4.1. $p = \frac{1}{2}$, $\mu = -1$, $\sigma = 1$, $X_0 = \frac{1}{4}$, $T = 3$

ε	$\mathsf{P}(X_T^n \le -\varepsilon) - \varepsilon$	$\mathsf{P}(X_T^n \le \varepsilon) + \varepsilon$	$\mathsf{P}(X_T = 0)$
$3 \cdot 10^{-6}$	.9679	.9738	[.9679, .9738]
$2 \cdot 10^{-6}$	.9700	.9738	[.9700, .9738]
$\boxed{10^{-6} = \varepsilon_n}$	.9738	.9738	[.9738, .9738]

TABLE 1.

Example 4.2. $p = \frac{1}{2}$, $\mu = +1$, $\sigma = 1$, $X_0 = \frac{1}{10}$, $T = 9$

ε	$\mathsf{P}(X_T^n \le -\varepsilon) - \varepsilon$	$\mathsf{P}(X_T^n \le \varepsilon) + \varepsilon$	$\mathsf{P}(X_T = 0)$
$2 \cdot 10^{-6}$	.8166	.8192	[.8166, .8192]
10^{-6}	.8174	.8192	[.8174, .8192]
$\boxed{5 \cdot 10^{-7} = \varepsilon_n}$	.8182	.8192	[.8182, .8192]

Table 2.

Example 4.3. $p = \frac{1}{2}$, $\mu = +1$, $\sigma = 1$, $X_0 = \frac{1}{4}$, $T = 9$

ε	$\mathsf{P}(X_T^n \le -\varepsilon) - \varepsilon$	$\mathsf{P}(X_T^n \le \varepsilon) + \varepsilon$	$\mathsf{P}(X_T = 0)$
$2 \cdot 10^{-6}$	.5960	.5970	[.5960, .5970]
10^{-6}	.5964	.5970	[.5964, .5970]
$\boxed{5 \cdot 10^{-7} = \varepsilon_n}$	.5970	.5970	[.5970, .5970]

Table 3.

Example 4.4. $p = \frac{1}{2}$, $\mu = +1$, $\sigma = 1$, $X_0 = 1$, $T = 9$

ε	$\mathsf{P}(X_T^n \le -\varepsilon) - \varepsilon$	$\mathsf{P}(X_T^n \le \varepsilon) + \varepsilon$	$\mathsf{P}(X_T = 0)$
$3 \cdot 10^{-6}$	.1344	.1348	[.1344, .1348]
$2 \cdot 10^{-6}$	.1346	.1348	[.1346, .1348]
$\boxed{10^{-6} = \varepsilon_n}$	.1346	.1348	[.1346, .1348]

Table 4.

Example 4.5. $p = \frac{3}{4}$, $\mu = +1$, $\sigma = 1$, $X_0 = \frac{1}{10}$, $T = 9$

ε	$\mathsf{P}(X_T^n \le -\varepsilon) - \varepsilon$	$\mathsf{P}(X_T^n \le \varepsilon) + \varepsilon$	$\mathsf{P}(X_T = 0)$
$3 \cdot 10^{-8}$	.6040	.6206	[.6040, .6206]
$5 \cdot 10^{-9}$	.6040	.6126	[.6126, .6206]
$\boxed{2.5 \cdot 10^{-9} = \varepsilon_n}$	.6180	.6206	[.6206, .6206]

Table 5.

Example 4.6. $p = \frac{3}{4}$, $\mu = +1$, $\sigma = 1$, $X_0 = \frac{1}{4}$, $T = 9$

ε	$\mathsf{P}(X^n_T \le -\varepsilon) - \varepsilon$	$\mathsf{P}(X^n_T \le \varepsilon) + \varepsilon$	$\mathsf{P}(X_T = 0)$
$3 \cdot 10^{-8}$	.3794	.3864	[.3794, .3864]
$5 \cdot 10^{-9}$	.3816	.3864	[.3816, .3864]
$\boxed{2.5 \cdot 10^{-9} = \varepsilon_n}$	.3838	.3864	[.3838, .3864]

Table 6.

Example 4.7. $p = \frac{3}{4}$, $\mu = +1$, $\sigma = 1$, $X_0 = 1$, $T = 9$

ε	$\mathsf{P}(X^n_T \le -\varepsilon) - \varepsilon$	$\mathsf{P}(X^n_T \le \varepsilon) + \varepsilon$	$\mathsf{P}(X_T = 0)$
$2 \cdot 10^{-8}$	.0752	.0790	[.0752, .0790]
10^{-8}	.0768	.0790	[.0768, .0790]
$\boxed{2.5 \cdot 10^{-9} = \varepsilon_n}$	.0782	.0790	[.0782, .0790]

Table 7.

Example 4.8. $p = \frac{3}{4}$, $\mu = -1$, $\sigma = 1$, $X_0 = \frac{1}{3}$, $T = 3$

ε	$\mathsf{P}(X^n_T \le -\varepsilon) - \varepsilon$	$\mathsf{P}(X^n_T \le \varepsilon) + \varepsilon$	$\mathsf{P}(X_T = 0)$
10^{-8}	.8536	.8803	[.8536, .0790]
$5 \cdot 10^{-9}$	.8700	.8803	[.8700, .0790]
$\boxed{2.5 \cdot 10^{-9} = \varepsilon_n}$	.8757	.8803	[.8757, .0790]

Table 8.

5. Existence and Uniqueness of solution in the CEV model

Proposition 5.1. *The Itô's equation* (1.1) *possesses a unique strong nonnegative solution.*

Proof. Define a sequence of processes indexed by integers i, $i > 1/X_0$, $(X^i_t)_{t\ge 0}$, such that X^i_t is a strong solutions to the following sde

$$dX^i_t = \mu X^i_t dt + \sigma(i^{-1} \vee |X^i_t|)^p dW_t, \ \ X^i_0 = X_0. \tag{5.1}$$

The diffusion coefficient $\sigma(i^{-1} \vee |x|)^p$ is Lipschitz continuous therefore X^i_t is the unique strong solution of (5.1). Set $\vartheta_i = \inf\{t : X^i_t = i^{-1}\}$. Note that for $t \le \vartheta_i$

$$X^{i+1}_t = X^i_t,$$

and it follows that $\vartheta_{i+1} > \vartheta_i$. A strong solution to (1.1) is constructed by a natural prolongation

$$X_{t\wedge\tau} := \sum_{i\ge n} X^i_{\vartheta_i} I_{\{\vartheta_i \le t < \vartheta_{i+1}\}}, \ \tau = \lim_{i\to\infty} \vartheta_i.$$

Finally, Yamada-Watanabe's theorem (see, e.g. [17, Rogers and Williams, p. 265]) guaranties the uniqueness of the strong solution of equation (1.1), because the Hölder parameter $p \ge \frac{1}{2}$. □

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